Announcements

1) Colloquium Wednesday,
2070 CB, 3–4
on Applied Math

2) HW – on #3, you
can use the formulae

$$(AB)^t = B^t A^t$$

$$(A^t)^t = A$$

On #4, your answers are most efficiently expressed in 2x2 "blocks":

$$A \in M_4(\mathbb{C})$$

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix}$$

with $A_{i,j}$'s in $M_2(\mathbb{C})$

What we know:

Every linear transformation between finite-dimensional vector spaces over $\mathbb{F}$ can be expressed as a matrix (basis dependent).

Any change-of-basis transformation may be represented as an invertible matrix.

Which matrices are the nicest to work with (while not being completely trivial)?

i.e., invertibility?

Diagonal matrices!

Example 1: (2x2 inverse)

Let $A \in M_2(\mathbb{C})$. Then if A is invertible and $A = (a_{i,j})_{i,j=1}^{2}$, then

$$A^{-1} = \frac{1}{a_{1,1}a_{2,2} - a_{1,2}a_{2,1}} \begin{bmatrix} a_{2,2} & -a_{1,2} \\ -a_{2,1} & a_{1,1} \end{bmatrix}$$

the determinant

Example 2: If

A is an $n \times n$ diagonal matrix and if A is invertible, then if

$$A = \begin{bmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_n \end{bmatrix},$$

$$A^{-1} = \begin{bmatrix} 1/a_1 & & & 0 \\ & 1/a_2 & & \\ & & \ddots & \\ 0 & & & 1/a_n \end{bmatrix}$$

Invertibility of a diagonal matrix is completely determined by whether the a_i's are zero or not.

All nonzero = invertible.

Otherwise, not invertible.

Definition: (matrix norm)

If V and W are both finite-dimensional normed linear spaces over $\mathbb{C}$ (or $\mathbb{R}$), then if $T: V \to W$ is linear and $\| \cdot \|_V$ and $\| \cdot \|_W$ are the norms on V and W, respectively

We can define

$$\boxed{\|T\| = \max_{\substack{\|x\|_V \leq 1 \\ x \in V}} \|Tx\|_W}$$

This is a norm on $\mathcal{L}(V,W)$.

"Max" is a consequence of finite-dimensionality.

We only care about

$\|\cdot\|_v = \|\cdot\|_w = \|\cdot\|_2$

in this class, although

$\|\cdot\|_1$ and $\|\cdot\|_\infty$ are

also interesting.

Observations: (diagonals in standard basis)

$A \in M_n(\mathbb{C})$,

$$A = \begin{bmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_n \end{bmatrix},$$

$$\boxed{\|A\| = \max_{1 \leq i \leq n} |a_i|}$$

To see this,

$$\|Ae_i\|_2 = \|a_i e_i\|_2$$
$$= |a_i| \, \|e_i\|_2 .$$

This says

$$\|A\| \geq \max_{1 \leq i \leq n} |a_i|$$

Conversely, if $\|h\|_2 = 1$,

$$h = \sum_{i=1}^{n} \beta_i e_i \text{ for some}$$

$\beta_i \in \mathbb{C}$, $1 \leq i \leq n$,

$$\|Ah\|_2^2 = \sum_{i=1}^{n} |a_i \beta_i|^2$$

$$= \sum_{i=1}^{n} |a_i|^2 |\beta_i|^2$$

$$\leq \max_{1 \leq i \leq n} |a_i|^2 \underbrace{\sum_{i=1}^{n} |\beta_i|^2}_{=1 \text{ since } \|h\|_2 = 1}$$

$$= \max_{1 \leq i \leq n} |a_i|^2$$

This shows

$$\|A\| \leq \max_{1 \leq i \leq n} \|Ah\|_2 .$$

Combining both inequalities,

$$\|A\| = \max_{1 \leq i \leq n} \|Ah\|_2 .$$

Example 3:

Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 8 \end{bmatrix}$.

Then

$$A = SDS^{-1}$$

with

$$D = \begin{bmatrix} 5+\sqrt{10} & 0 \\ 0 & 5-\sqrt{10} \end{bmatrix}$$

$$S = \begin{bmatrix} -3-\sqrt{10} & 1 \\ 1 & 3+\sqrt{10} \end{bmatrix}$$

We get that

$$\boxed{\begin{gathered} \|A\| = \|D\| = 5+\sqrt{10} \\ A^{-1} = S\,D^{-1}S^{-1},\quad D^{-1} = \begin{bmatrix} \frac{1}{5+\sqrt{10}} & 0 \\ 0 & \frac{1}{5-\sqrt{10}} \end{bmatrix} \end{gathered}}$$

First Question: When

can we write

$A \in M_n(\mathbb{C})$ as

$$\boxed{A = SDS^{-1}}$$

for some diagonal matrix D?

Definition: (eigenvalues, eigenvectors) A complex number λ is said to be an eigenvalue of $A \in M_n(\mathbb{C})$ if

$\exists$ $v \in \mathbb{C}^n$, $v \neq (0, 0, - 0)$

$$\boxed{Av = \lambda v}$$

The vector v is called an eigenvector for the eigenvalue λ.

Note: λ could be zero, but v must always be nonzero!

Theorem: (eigenspace)

Let λ be an eigenvalue of $A \in M_n(\mathbb{C})$. Then

$$\{v \in \mathbb{C}^n \mid Av = \lambda v\} = W_\lambda$$

is a subspace of $\mathbb{C}^n$, called the eigenspace associated to λ.

proof: Subspace test!

1) $\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in W_\lambda$ since

$$A\left(\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= \lambda \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

2) If $v \in W_\lambda$, $\alpha \in \mathbb{C}$,

$$A(\alpha v) = \alpha A(v)$$
$$= \alpha \lambda v$$
$$= \lambda (\alpha v)$$

$\Rightarrow \alpha v \in W_\lambda$

3) If $w, v \in W_\lambda$,

$$A(w+v) = A(w) + A(v)$$

$$= \lambda w + \lambda v$$

$$= \lambda (w+v)$$

$$\Rightarrow \quad w+v \in W_\lambda$$

$\square$

Equivalence

If $Av = \lambda v$ $(v \neq 0_V)$,

then $Av - \lambda v = 0_V$.

This is equivalent to

$$(A - \lambda I_n)v = 0_V$$

This says

$\ker(A - \lambda I_n) \neq \{0_V\}$,

so $A - \lambda I_n$ is not

invertible. Conversely,

if $A - \lambda I_n$ is not

invertible, then $\exists\, v \in \mathbb{C}^n$,

$v \neq 0_V$, $(A - \lambda I_n)v = 0_V$

$\Rightarrow \lambda$ is an eigenvalue for A.

In order to quickly tell whether $A - \lambda I_n$ is invertible, we will develop the determinant.